Comp. & Maths. with Appls., Vol. 2, pp. 255–265. Pergamon Press 1976. Printed in Great Britain

# A RECURSIVE RADIX CONVERSION FORMULA AND ITS APPLICATION TO MULTIPLICATION AND DIVISION

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Communicated by E. Y. Rodin

(*Received March* 1976)

**Abstract**—A recursive formula for number conversion from one radix representation to another radix representation is presented. This formula differs from the existing ones in two major aspects. First, it utilizes a digit shift technique which provides faster accumulation of higher significant digits in the final result. Second, it is suitable for parallel computation, so that the length of time necessary for number conversion can be shortened. Thus the longer the digit number, the more appreciation in conversion time-saving will result.

Applications of the recursive formula are studied in multiplication and division for negative radix numbers as well as for positive radix numbers. The multiplication and division presented here are especially useful for computations of $\eta$ word precision, since successive bit-carrying propagation to the most significant digit hardly ever occurs.

## 1. INTRODUCTION

Internal number representation in a digital computer is usually different from decimal representation which has been utilized in our daily lives for centuries. Thus, a conversion process from decimal to internal representation and vice versa is always necessary in order to perform a computation within a digital computer. Binary representation is employed in most digital computers for hardware simplicity, so that the conversions from decimal to binary (or its power representation $2^i$ for $i = 2, 3, \ldots$) before a computation and from binary to decimal after the computation take place internally. The other number representations (ternary and negative binary) have been used in a few digital computers that were built in eastern Europe.

Negative radix number representation[4, 13] as well as positive radix number representation[8] has been discussed, and computer designs utilizing negative number representation have been presented in [2, 5, 7, 15]. Arithmetic for negative radix numbers has been discussed in [2] and [10], and improvements of multiplication and division in negative radix numbers have been investigated by Agrawal, Houselander, and Sanker *et al.*[1, 6, 11]. Similarly, faster computation schemes for multiplication and division of positive radix numbers have been studied by Baker, Robertson, Tocher and Wilson *et al.*[3, 9, 12, 14].

The recursive conversion formula here is derived mathematically. Each digit conversion process employs a one-digit-shift operation towards the most significant digit. It provides one-digit-lengthless computation at each conversion step when compared to a well known method[8]. Therefore, the recursive process could save a processing time of the final $m$ digit addition that is needed in the well known method, where $m$ is a digit number of the converted number. The conversion process utilizes two intermediate accumulation registers. Each intermediate accumulation can be carried out independently, so that the process is suitable for parallel computation. Thus the conversion process is advantageous for a longer digit conversion such as double word precision (or $\eta$ word precision where $\eta \geq 2$).

Applications of the recursive conversion formula to multiplication and division in both positive and negative radix representations are studied. Multiplication can be performed as the final digit conversion process in the recursive conversion formula. Division becomes a reverse computation form an $m^{\text{th}}$ step to the $(m-1)^{\text{th}}$ step in the recursive conversion formula.

## 2. A RECURSIVE CONVERSION FORMULA

Any positive number $N$ can be represented in a radix number system $D$ (called the present radix) as follows:

$$N = \eta_m D^m + \eta_{m-1} D^{m-1} + \eta_{m-2} D^{m-2} + \cdots + \eta_1 D^1 + \eta_0 D^0 \tag{2.1}$$

where $m+1$ is the number of digits of $N$ and $\eta_i$ is a digit, such that $0 \leq \eta_i < D$ for $i = 0, 1, 2, \ldots, m$. The number $N$ can also be represented in another radix number system $B$ (called the target radix) where $D = B + P$ $(\neq 0, 1)$. $P$ is a positive or negative integer, so (2.1) becomes:

$$N = \eta_m(B+P)^m + \eta_{m-1}(B+P)^{m-1} + \cdots + \eta_1(B+P)^1 + \eta_0(B+P)^0. \tag{2.2}$$

By expanding $(B+P)^i$ for $i = 1, 2, 3, \ldots, m$, we get the following terms:

$$\begin{aligned}
&\eta_m\left\{B^m + \binom{m}{1}B^{m-1}P + \cdots + \binom{m}{m-1}BP^{m-1} + P^m\right\}\\
&+\eta_{m-1}\left\{B^{m-1} + \binom{m-1}{1}B^{m-2}P + \cdots + \binom{m-1}{m-2}BP^{m-2} + P^{m-1}\right\}\\
&+\eta_{m-2}\left\{B^{m-2} + \binom{m-2}{1}B^{m-3}P + \cdots + \binom{m-2}{m-3}BP^{m-2} + P^{m-2}\right\}\\
&+\cdots\\
&\vdots\\
&+\eta_2\{B^2 + 2BP + P^2\}\\
&+\eta_1\{B + P\}\\
&+\eta_0.
\end{aligned} \tag{2.3}$$

When the term $B^i$ for $i = 0, 1, 2, \ldots, m$ is summed, (2.3) becomes (2.4) as shown below:

$$\begin{aligned}
&B^m\eta_m\\
&+B^{m-1}\left\{\eta_m\binom{m}{1}P + \eta_{m-1}\right\}\\
&+B^{m-2}\left\{\eta_m\binom{m}{2}P^2 + \eta_{m-1}\binom{m-1}{1}P + \eta_{m-2}\right\}\\
&+B^{m-3}\left\{\eta_m\binom{m}{3}P^3 + \eta_{m-1}\binom{m-1}{2}P^2 + \eta_{m-2}\binom{m-2}{1}P + \eta_{m-3}\right\}\\
&+\cdots\\
&\vdots\\
&+B^1\left\{\eta_m\binom{m}{m-1}P^{m-1} + \eta_{m-1}\binom{m-1}{m-2}P^{m-2} + \eta_{m-2}\binom{m-2}{m-3}P^{m-3} + \cdots + \eta_1\right\}\\
&+B^0\{\eta_mP^m + \eta_{m-1}P^{m-1} + \eta_{m-2}P^{m-2} + \cdots + \eta_2P^2 + \eta_1P + \eta_0\}.
\end{aligned} \tag{2.4}$$

In the target radix, $T$ is another representation for $N$. Each quantity inside the braces is evaluated and summed; therefore,

$$T = d_lB^l + d_{l-1}B^{l-1} + d_{l-2}B^{l-2} + \cdots + d_1B^1 + d_0B^0 \tag{2.5}$$

where $0 \leq d_j < B$ for $j = 0, 1, 2, \ldots, l$.

By comparing (2.4) and (2.5), it is clear that the quantity of (2.4) becomes the quantity of (2.5) after converting each term of (2.4) in the target radix. The summation can be done systematically in the manner presented in the following two theorems.

THEOREM 1

Let the $j^{\text{th}}$ term of (2.4) be:

$$\begin{aligned}
R_j(i) = \eta_m\binom{i}{j}P^i + \eta_{m-1}\binom{i-1}{j-1}P^{i-1} + \cdots + \eta_{m-j+2}\binom{i-j+3}{2}P^2\\
+\eta_{m-j+1}\binom{i-j+2}{1}P + \eta_{m-i}
\end{aligned}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$, and $R_0(i) = \eta_m$ for $i = 0, 1, 2, \ldots, m$. If

$$r_i = \sum_{k=0}^{i-1} R_k(i) B^{i-1-k} \quad \text{for} \quad i = 1, 2, \ldots, m$$

$$r_0 = 0$$

then

$$r_i = t_{i-1} + r_{i-1}P \quad \text{for} \quad i = 1, 2, \ldots, m$$

where

$$t_{i-1} = r_{i-1}B + R_{i-1}(i-1).$$

In order to prove Theorem 1, we need the following two lemmas:

LEMMA 1

$$1 + \binom{i}{1} = \binom{i+1}{1}$$

*Proof*

$$1 + \binom{i}{1} = 1 + i = \binom{i+1}{1} \qquad \blacksquare$$

LEMMA 2

$$\binom{i}{j} + \binom{i}{j+1} = \binom{i+1}{j+1}$$

*Proof*

$$\begin{aligned}
\binom{i}{j} + \binom{i}{j+1} &= \frac{i(i-1)(i-2)\cdots(i-j+1)}{j(j-1)\cdots 3\cdot 2\cdot 1} + \frac{i(i-1)(i-2)\cdots(i-j+1)(i-j)}{(j+1)j(j-1)\cdots 3\cdot 2\cdot 1} \\
&= \frac{(j+1)i(i-1)(i-2)\cdots(i-j+1) + i(i-1)(i-2)\cdots(i-j+1)(i-j)}{(j+1)j(j-1)\cdots 3\cdot 2\cdot 1} \\
&= \frac{(j+1+i-j)i(i-1)(i-2)\cdots(i-j+1)}{(j+1)j(j-1)\cdots 3\cdot 2\cdot 1} \\
&= \frac{(i+1)i(i-1)(i-2)\cdots(i-j+1)}{(j+1)j(j-1)\cdots 3\cdot 2\cdot 1}
\end{aligned}$$

The last term becomes $\binom{i+1}{j+1}$.

A proof of the reverse course is very much the same as the proof presented above; therefore, it is omitted. $\blacksquare$

*Proof of Theorem 1*

Since $m$ is a parameter of the theorem, each term in the braces of each row of (2.4) can be represented in a recursive formula.

From Lemma 1, the second row brace can be written in the following recursive formula:

$$R_1(i) = \eta_m \binom{i}{1} P + \eta_{m-1} = \eta_m \binom{i-1}{1} P + \eta_{m-1} + \eta_m P = R_1(i-1) + R_0(i-1)P \tag{2.6}$$

where $m \geq i > 1$.

Again from Lemmas 1 and 2, the third row brace can be written in the next recursive formula:

$$\begin{aligned} R_2(i) &= \eta_m\binom{i}{2}P^2 + \eta_{m-1}\binom{i-1}{1}P + \eta_{m-2} \\ &= \eta_m\left\{\binom{i-1}{1} + \binom{i-1}{2}\right\}P^2 + \eta_{m-1}\left\{1 + \binom{i-2}{1}\right\}P + \eta_{m-2} \end{aligned} \tag{2.7}$$

where $m \geq i > 2$

The right-hand side of (2.7) can be arranged as follows:

$$\left\{\eta_m\binom{i-1}{1}P + \eta_{m-1}\right\}P + \eta_m\binom{i-1}{2}P^2 + \eta_{m-1}\binom{i-2}{1}P + \eta_{m-2} = R_2(i-1) + R_1(i-1)P. \tag{2.8}$$

The first term of (2.8) is the right-hand side of (2.6) except its argument $i$ is replaced by $i-1$, and the rest of (2.8) becomes a recursive form $R_2(i-1)$ of the left-hand side $R_2(i)$. Similarly from Lemmas 1 and 2, the fourth row brace can be rewritten as follows:

$$\begin{aligned} R_3(i) &= \eta_m\binom{i}{3}P^3 + \eta_{m-1}\binom{i-1}{2}P^2 + \eta_{m-2}\binom{i-2}{1}P + \eta_{m-3} \\ &= \eta_m\left\{\binom{i-1}{2} + \binom{i-1}{3}\right\}P^3 + \eta_{m-1}\left\{\binom{i-2}{1} + \binom{i-2}{2}\right\}P^2 + \eta_{m-2}\left\{1 + \binom{i-3}{1}\right\}P + \eta_{m-3} \end{aligned}$$

where $m \geq i > 3$. (2.9)

The right-hand side of (2.9) is rearranged as shown below:

$$\begin{aligned} &\left\{\eta_m\binom{i-1}{2}P^2 + \eta_{m-1}\binom{i-2}{1}P + \eta_{m-2}\right\}P + \eta_m\binom{i-1}{3}P^3 + \eta_{m-1}\binom{i-2}{2}P^2 + \eta_{m-2}\binom{i-3}{1}P + \eta_{m-3} \\ &= R_3(i-1) + R_2(i-1)P. \end{aligned} \tag{2.10}$$

The first term of (2.10) is (2.7) except its argument $i$ is replaced by $i-1$, and the rest of (2.10) becomes a recursive form $R_3\ (i-1)$ of the left-hand side $R_3(i)$.

Generally, let the $j^{\text{th}}$ row of (2.4) be represented in the following equation:

$$\begin{aligned} R_{j-1}(i) &= \eta_m\binom{i}{j-1}P^{j-1} + \eta_{m-1}\binom{i-1}{j-2}P^{j-2} + \cdots + \eta_{m-j+2}\binom{i-j+2}{1}P + \eta_{m-j+1} \\ &= \eta_m\left\{\binom{i-1}{j-2} + \binom{i-1}{j-1}\right\}P^{j-1} + \eta_{m-1}\left\{\binom{i-2}{j-3} + \binom{i-2}{j-2}\right\}P^{j-2} + \cdots \\ &\quad + \eta_{m-j+2}\left\{1 + \binom{i-j+1}{1}\right\}P + \eta_{m-j+1} \\ &= \left\{\eta_m\binom{i-1}{j-2}P^{j-2} + \eta_{m-1}\binom{i-2}{j-3}P^{j-3} + \cdots + \eta_{m-j+2}\right\}P \\ &\quad + \eta_m\binom{i-1}{j-1}P^{j-1} + \eta_{m-1}\binom{i-2}{j-2}P^{j-2} + \cdots + \eta_{m-j+2}\binom{i-j+1}{1}P + \eta_{m-j+1} \\ &= R_{j-1}(i-1) + R_{j-2}(i-1)P \qquad m \geq i > j-1. \end{aligned} \tag{2.11}$$

Then it is assumed that the first term is equal to the $(j+1)^{\text{th}}$ row recursive formula except for its argument, and the rest becomes a recursive form $R_{j-1}(i-1)$.

The $(j+1)^{\text{th}}$ row of (2.4) is:

$$R_j(i) = \eta_m\binom{i}{j}P^j + \eta_{m-1}\binom{i-1}{j-1}P^{j-1} + \cdots + \eta_{m-j+2}\binom{i-j+3}{2}P^2$$
$$+ \eta_{m-j+1}\binom{i-j+2}{1}P + \eta_{m-j}$$
$$= \eta_m\left\{\binom{i-1}{j-1} + \binom{i-1}{j}\right\}P^j + \eta_{m-1}\left\{\binom{i-2}{j-2} + \binom{i-2}{j-1}\right\}P^{j-1} + \cdots$$
$$+ \eta_{m-j+2}\left\{\binom{i-j+2}{1} + \binom{i-j+2}{2}\right\}P^2 + \eta_{m-j+1}\left\{1 + \binom{i-j-1}{1}\right\}P + \eta_{m-j}$$
$$= \left\{\eta_m\binom{i-1}{j-1}P^{j-1} + \eta_{m-1}\binom{i-2}{j-2}P^{j-2} + \cdots + \eta_{m-j+2}\binom{i-j+2}{1}P + \eta_{m-j+1}\right\}P$$
$$+ \eta_m\binom{i-1}{j}P^j + \eta_{m-1}\binom{i-2}{j-1}P^{j-1} + \cdots + \eta_{m-j+2}\binom{i-j+2}{2}P^2 + \eta_{m-j+1}P$$
$$+ \eta_{m-j} = R_j(i-1) + R_{j-1}(i-1)P \qquad m \geq i > j. \tag{2.12}$$

The first term of (2.12) is the same as (2.11) except for its argument $i$, and the rest becomes a recursive form $R_j(i-1)$ so

$$R_j(i) = R_j(i-1) + R_{j-1}(i-1)P.$$

*Note*: $R_j(i)$ is only defined when $i \geq j$.

From (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12), we have:

$$R_j(i) = R_j(i-1) + R_{j-1}(i-1)P \tag{2.13}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$.
Since

$$r_{i-1} = \sum_{k=0}^{i-2} R_k(i-1)B^{i-2-k}$$

therefore

$$t_{i-1} = r_{i-1}B + R_{i-1}(i-1) = \left\{\sum_{k=0}^{i-2} R_k(i-1)B^{i-2-k}\right\}B + R_{i-1}(i-1)$$
$$= \sum_{k=0}^{i-1} R_k(i-1)B^{i-1-k}. \tag{2.14}$$

From (2.13) and the hypothesis,

$$r_i = \sum_{k=0}^{i-1} R_k(i)B^{i-1-k} = R_0(i)B^{i-1} + \sum_{k=1}^{i-1} \{R_k(i-1) + R_{k-1}(i-1)P\}B^{i-1-k}$$
$$= \sum_{k=0}^{i-1} R_k(i-1)B^{i-1-k} + \sum_{k=1}^{i-1} R_{k-1}(i-1)PB^{i-1-k}.$$

From (2.14) and by changing the range $1 \to i-1$, of the running number $k$ to $0 \to i-2$ in the second term,

$$r_i = t_{i-1} + \sum_{k=0}^{i-2} R_k(i-1)PB^{i-2-k} = t_{i-1} + r_{i-1}P. \tag{2.15}$$

We have completed the proof of the theorem. ■

Further, we derive another recursive formula.

Let

$$R_i(i) = s_i = \sum_{k=0}^{i} \eta_{m-k} P^{i-k}$$

so that

$$s_{i+1} = \sum_{k=0}^{i+1} \eta_{m-k} P^{i+1-k} = \left(\sum_{k=0}^{i} \eta_{m-k} P^{i-k}\right) P + \eta_{m-i-1} = s_i P + \eta_{m-(i+1)}. \tag{2.16}$$

From the above, we have the following recursive radix conversion theorem.

THEOREM 2

Let a number $N$ in a present radix $D$ be:

$$N = \eta_m D^m + \eta_{m-1} D^{m-1} + \eta_{m-2} D^{m-2} + \cdots + \eta_1 D^1 + \eta_0 D^0$$

where $0 \le \eta_i < D$ for $i = 0, 1, 2, \ldots, m$.
If $s_0 = \hat{\eta}_m$, $r_0 = 0$, $t_0 = s_0$, and

$$s_i = s_{i-1} P + \hat{\eta}_{m-i} \tag{2.17}$$

$$r_i = t_{i-1} + r_{i-1} P \tag{2.18}$$

$$t_i = r_i B + s_i \tag{2.19}$$

for $i = 1, 2, 3, \ldots, m$, then the converted number $T$ for $N$ in target radix $B$ (where $D = B + P$ and where $|B|$ and $|D| \ge 2$) is:

$$T = t_m.$$

*Note*: (1) $\hat{\eta}_i$ for $i = 0, 1, 2, \ldots, m$ is the converted digit of $\eta_i$ in the target radix $B$ where $B \le \eta_i < D$, otherwise $\hat{\eta}_i = \eta_i$.

(2) All arithmetic must be performed in the target radix $B$.

*Proof*

We have derived the recursive radix conversion formula which consists of (2.17), (2.18), and (2.19) from (2.1) and (2.2). By starting with the initial condition in the recursive formula, a successive substitution into the recursive formula shall reach the converted number $T$ for $N$ in the target radix as shown in (2.3), (2.4), (2.5) and Theorem 1. ∎

EXAMPLE 1

Convert $5250_9$ to $XXXX_{10}$, $D = 9$, $B = 10$ and $P = -1$.

| $i$ | $\hat{\eta}_{m-i}$ | $s_i$ | $r_i$ | $t_i$ |
|---|---|---|---|---|
| 0 | 5 | 5 | 0 | 5 |
| 1 | 2 | $5(-1)+2=-3$ | $5+0(-1)=5$ | $50+(-3)=47$ |
| 2 | 5 | $(-3)(-1)+5=8$ | $47+5(-1)=42$ | $420+8=428$ |
| 3 | 0 | $8(-1)+0=-8$ | $428+42(-1)=386$ | $3860+(-8)=3852$ |

$\therefore T = 3852_{10}$.

Since the derivation process of the recursive radix conversion formula assumes no special condition in the radix, the formula can be applied on a conversion between any radices except when $|B|$ and $|D|$ are less than 2. In order to demonstrate a conversion from $D = 10$ to $B = -10$, we introduce an addition table for a $-10$ radix.

There are two tables nominated as Table A and Table B; each table is symmetrical, so that it is

Table A

| A | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 |
| 7 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

Apply table A next — Apply table B next

Apply table A next — Apply table B next

Table B

| B C | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
| 3 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 |
| 4 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 |
| 5 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 |
| 6 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 |
| 7 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 |
| 8 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 9 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Apply table A next — Apply table B next

Apply table A next — Apply table B next

not necessary to distinguish the addend and the addendum. Addition starts at the least significant digit when using Table A. The result of the addition is shown in an entry intersected by a row and a column. There are two areas in each table. One indicates that Table A must be used in the next higher digit addition, and the other shows that Table B must be applied. The $C$ in the upper left corner of Table B indicates a carried bit. For another negative radix, a similar table can be constructed.

EXAMPLE 2

Convert $3749_{10}$ to $XXXX_{-10}$, where $B = -10$, $D = 10$, and $P = 20_{10} = 180_{-10}$.

| $i$ | $\hat{\eta}_{m-i}$ | $s_i$ | $r_i$ | $t_i$ |
|---|---|---|---|---|
| 0 | 3 | 3 | 0 | 3 |
| 1 | 7 | $3 \cdot 180 + 7$ <br> $= 140 + 7$ <br> $= 147$ | 3 | $30 + 147 = 177$ |
| 2 | 4 | $147 \cdot 180 + 4$ <br> $= 19460 + 4$ <br> $= 19464$ | $3 \cdot 180 + 177$ <br> $= 140 + 177$ <br> $= 117$ | $1170 + 19464$ <br> $= 434$ |
| 3 | 9 | $19464 \cdot 180 + 9$ <br> $= 34920 + 9$ <br> $= 34929$ | $117 \cdot 180 + 434$ <br> $= 18060 + 434$ <br> $= 18494$ | $184940 + 34929$ <br> $= 17869$ |

$$\therefore T = 17869_{-10}.$$

The product of $147 \cdot 8$ is found as follows:

$$147 + 147 = 274, \quad 274 + 274 = 348, \quad 348 + 348 = 676$$

so that $147 \cdot 180 = 14700 + 6760 = 19460$. The other multiplications in the conversion steps can be computed similarly. Alternatively, these multiplications can be done by the multiplication scheme presented in the following section.

## 3. APPLICATIONS

Multiplication and division become a one-digit-shift operation to the left and to the right, if the multiplicand and the dividend are represented in the radix of the multiplier and the divisor, respectively.

Consider the multiplication $A \cdot B$ wheere $A$ and $B$ are represented in a radix $D$. The multiplier $B$ is regarded as a target radix in a radix conversion process. Suppose the converted number of the multiplicand $A$ from the radix $D$ to the radix $B$ is known. Multiplication becomes a one-digit-shift operator to the left by adding a tailing zero as the least significant digit. Since the numbers $A$ and $B$ are represented originally in the radix $D$, the result of a multiplication must be represented in the radix $D$. Processing the reverse conversion course, namely from radix $B$ to radix $D$, the original number $A$ appears in a single step before the final digit conversion which obtains the multiplication result. The last digit is the zero introduced by the shift operation. Thus, multiplication can be performed as the final digit conversion process in the recursive conversion formula.

A similar argument for division from the recursive conversion formula can be developed like multiplication. However, there are two differences—the direction of the one-digit-shift operation is to the right instead of to the left and the original dividend $A$ in the division $A/B$ appears in one more conversion step after the final digit conversion to obtain the quotient in the reverse conversion course (from radix $B$ to radix $D$). Thus, division becomes a process to find the final converted digits from the dividend $A$ represented in the radix $D$, in other words, a reverse computation from an $m^{\text{th}}$ step to the $(m-1)^{\text{th}}$ step in the recursive conversion formula. Precise descriptions of multiplication and division from the recursive formula follow.

*Multiplication*

Consider $A \cdot B$ where $A$ and $B$ are represented in a radix $D$ and $D = B + P_f$. The multiplier $B$ is deemed as the target radix in the shift operation for multiplication. Also, consider the reverse conversion course from $B$ to $D$ where $B = D + P_r$ and $P_r = -P_f$. Thus, $t_{m-1} = A$ and $\hat{\eta}_0 = 0$ (obtained by the one-digit-left shift at the least significant digit). The values for $s_{m-1}$ and $r_{m-1}$ can be obtained by splitting $t_{m-1}$. Digit splitting of $t_{m-1}$ is arbitrary, and any digit split does not make a difference in the final $m^{\text{th}}$ conversion step.

EXAMPLE 3

Find the product of $675 \times 39$, where $D = 10$, $B = 39$, $P_r = 29$, and $t_{m-1} = 675$. Let $s_{m-1} = 5$ and $r_{m-1} = 67$, so $s_m = 5 \cdot 29 + 0 = 145$ and $r_m = 67 \cdot 29 + 675 = 2618$. Thus, $T = t_m = 26180 + 145 = 26325$.

Each application of this process reduces not only the number of multiplication digits of the multiplicand but also uses a less digit number manipulation (or the shift operation on $r_m$) in the final result $t_m$. Furthermore, multiplications, $s_{m-1}P$ and $r_{m-1}P$, can be carried out by another application of this process successively until all of these multiplications become one digit by one digit multiplication. When multiplicand digits are bisected repeatedly, the entire multiplication could be accomplished faster with parallel operations. The most ideal parallel operation is a process exercised in the fashion of a balanced binary tree. This process is stated in the following corollary.

COROLLARY 3

If digits of the multiplicand $A = t_{m-1}$ are properly split into $s_{m-1}$ and $r_{m-1}$ based on radix $D$ then the multiplication $A \cdot B$ in radix $D$ is accomplished as the final digit conversion process in the recursive formula where $P_r = B - D$ and $\hat{\eta}_0 = 0$. A proof is omitted.

*Division*

A similar process like multiplication exists in division. Consider $A/B$ where $A$ and $B$ are represented in a radix $D$ and $D = B + P_f$. The dividend $B$ is regarded as the target radix. The reverse conversion course from $B$ to $D$ where $B = D + P_r$ and $P_r = -P_f$ at the $m^{\text{th}}$ step is as follows.

$$\begin{array}{lll} s_{m-1} & r_{m-1} & t_{m-1} = r_{m-1}D + s_{m-1} \\ s_m = s_{m-1}P_r + R & r_m = r_{m-1}P_r + r_{m-1}D + s_{m-1} & t_m = r_mD + s_m = A \end{array}$$

where $R$ is the remainder (or the last digit in the target radix) and $t_{m-1}$ is the quotient.

By substituting $s_m$ and $r_m$ into the last equation $t_m = A$ in the $m^{\text{th}}$ step, the resulting equation is:

$$(r_{m-1}P_r + r_{m-1}D + s_{m-1})D = A - s_{m-1}P_r - R \tag{3.1}$$

where $A$, $D$ and $P_r$ are known variables but $s_{m-1}$, $r_{m-1}$ and $R$ are unknown variables. Splitting the quantity $A$ into $r_m$ and $s_m$ is arbitrary like multiplication, so that a splitting of $A$ is assumed as

$r_{m-1}$ becomes zero. Equation (3.1) becomes:

$$Dx = A - Px - R$$

where $x$ and $P$ denote $s_{m-1}$ and $P_r$, respectively.

Since $R$ is the remainder, $0 \leq R < B$. A range of the value $x$ is determined from two equations.

$$Dx = -Px + A \tag{3.2.a}$$

$$Dx = -Px + A - D. \tag{3.2.b}$$

Equation (3.2) can be solved in an iteration scheme using the following two equations.

$$y_{i+1} = -Px_i + A \tag{3.3.a}$$

$$x_{i+1} = y_{i+1}/D. \tag{3.3.b}$$

The sequences, $y_1 = A$, $y_2 = A(1 - P/D)$, $y_3 = A(1 - P/D + P^2/D^2), \ldots$ and $x_1 = A/D$, $x_2 = A(1 + P/D)/D$. $x_3 = A(1 - P/D + P^2/D^2)/D, \ldots$ for $i = 1, 2, 3, \ldots$ are obtained from (3.3) by starting with $x_0 = 0$. Hence the solution of (3.2.a) is:

$$x = A\{1 - P/D + (P/D)^2 - (P/D)^3 + \cdots\}/D. \tag{3.4}$$

*Note.* The operation $(1/D)^i$ is just a digit shift operation to the right by the $i$ digits, which means no division is involved.

A schematic iterative process of the solution of the simultaneous equations $y = -Px + A$ and $y = Dx$ is depicted in the Fig. 1.

If $|P/D| < 1$ then (3.4) is convergent. The quantity $x$ is $(A/D)/\{1 - (-P/D)\} = A/(D + P) = A/B$. If a pseudoradix $D'$ is chosen as $D' = D^k$ such that $|D^k| > |B|$ with the smallest integer $k$ for $k = 1, 2, 3, \ldots$ where $D$ is the present radix in a division $A/B$, then the convergency condition $|P/D'| < 1$ is always satisfied. However when $1 > |P/D'| > \frac{1}{2}$, the convergency of (3.4) is slower. The following acceleration factor $\alpha$, such that $\alpha$ is the largest integer $m^*$ among $m \cdot |B| < |D'|$ for $m = 1, 2, 3, \ldots$, provides a faster convergency of the division $A \cdot \alpha/(B \cdot \alpha)$. The truncation error $|\epsilon|$ at the $i^{\text{th}}$ term of (3.4) is $|A/B| \cdot |P/D|^i$.

*Note*: When the signs of $D$ and $B$ are different, $|P|$ can be greater than $|D|$ (or $|D'|$). This difficult can be circumvented by multiplying minus one $(-1)$ to the dividend and the divisor for division and to the multiplicand and the multiplier for multiplication.

Example 4

Consider 1868/183 in the present radix $D = -10$ (decimal) where $A = 1868 = -252$ (decimal) and $B = 183 = 23$ (decimal). Take $D' = 100$, then evaluate for $x$ in (3.4) with $P/D = 83/100$ converging slowly. Choose $\alpha = 4$, or $23 \times 4 < 100$ (in decimal), and compute $1868 + 1868 = 1516$, $1516 + 1516 = 1012$ so $1868 \times 4 = 1012$ and similarly $183 \times 4 = 112$ from the addition table. When

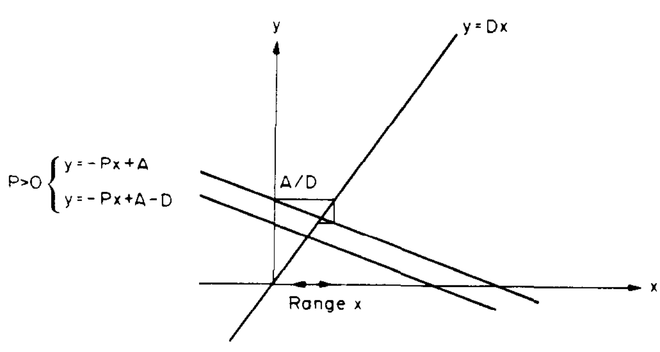


Fig. 1.

$P/D'$ is 12/100, (3.4) becomes:

$$\begin{aligned}1012/112 &= 1012\{1-12/100+(12/100)^2-\cdots\}/100\\ &= 1012\{10000-1200+144-\cdots\}/1000000\\ &= 1012\{10144+800-\cdots\}/10^6\\ &\doteqdot 1012\cdot 10944/10^6 = 29051128/10^6.\end{aligned}$$

The multiplications $12\cdot 12$ and $10944\cdot 1012$ are performed using the recursive radix conversion formula as shown in Example 3, where $A=12, B=12, D=10$, and $P=2$ and $A=10944, B=1012$, $D'=1000$, and $P=12$, respectively. Detailed steps of the latter are presented below.

$$\begin{array}{lll} s_{m-1}=944 & r_{m-1}=10 & t_{m-1}=10944\\ s_m = 944\cdot 12+0 & r_m = 10\cdot 12+10944 & t_m = 29064000+7128\\ \quad =7128 & \quad = 120+10944 & \quad =29051128\\ & \quad =29064 & \end{array}$$

where $944\cdot 12$ is performed again using the recursive formula. The number of terms ($i$) needed for a division can be determined by $|A/B|\cdot|P/D|^i$ in order to satisfy a given accuracy. The error caused by truncated terms in (3.4) is $|-252/23|\cdot|-8/100|^3 \simeq 0.01895$ in example 4 so the quotient is 29 ($=t_{m-1}$). The remainder $R$ is found from $s_m = s_{m-1}P_r + R$, where $r_m = t_{m-1} = s_{m-1} = 29$, $P_r = 83$, $r_{m-1}=0$, and $D'=100$. We find $s_m = t_m - r_m\cdot D' = 1868-2900 = 968$. After a simple computation, $R$ is determined as $968-29\cdot 83 = 968-967 = 968+1253 = 1$. Since the signs of the dividend and the remainder are different, the quotient 29 is an over-subtracted value from the dividend. A correction must be performed to obtain the right quotient $29+1=10$, then the new remainder is determined as $R = 868-830 = 38$ from $s_m = 868$, $s_{m-1}=10$, and $P_r = 83$.

Negation of a number in the radix $-10$ is accomplished by accumulating each digit complement defined as $c(0)=0$ and $c(\eta) = 1\eta_d$, where $0<\eta\le 9$ and $\eta_d$ is the absolute difference between $\eta$ and 10 (decimal). For example, $-128$ (in the radix $-10$) $= 1900+180+12 = 92$.

It is necessary to use more terms or to obtain a smaller truncation error in (3.4) in order to find the floating point number quotient of a division. For example the first seven terms must be evaluated for $x$ in (3.4) in order to satisfy $|\epsilon|\le 10^{-8}$ in Example 4.

This division process is stated in the following corollary.

COROLLARY 4

If $|P/D'|<1$ where $P=B-D'$, $B$ is the divisor and $D'$ is a pseudoradix then the division $A/B$ in radix $D$ is performed by evaluating the equation (3.4).

The truncation error by the $i^{\text{th}}$ terms of (3.4) is determined by $|A/B|\cdot|P/D'|^i$.

A proof is omitted.

Another deterministic division process derived from the recursive formula is presented in the appendix.

## 4. REMARKS

The recursive radix conversion formula is presented. The conversion process of the recursive formula could shorten the conversion processing time when compared with a well known method. First, when the one-digit-shift operation at each digit conversion is performed, there is one-digit-less computation. Second, parallel computation of the intermediate accumulations is appropriate and shortens the conversion time. Thus, the longer the digit number, the more appreciation in conversion time-saving will result. Since a negative radix is just another radix, the recursive conversion process can be applicable for radix conversion from a positive radix to a negative radix and vice versa.

The applications of the recursive conversion process to multiplication and division are discussed. In multiplication, a long digit multiplication could be performed with parallel computations of partial multiplications utilizing successive applications of the final stage of the recursive conversion. Furthermore, the digit number of $s_m$ is generally far less than the digit number of $r_m$; therefore, bit-carrying propagation to reach the most significant digit position never occurs. Hence an assembly process of the partial multiplication results (computed by the parallel operations) can be furnished easily so that multiplication could be accomplished faster.

A division can be performed by a set of operations, or additions, digit shifts, and multiplications in which the operational sequence of these is derived from the recursive conversion formula. Moreover, this division process is a deterministic process since it does not require sign bit comparison between dividend and divisor in the course to determine each digit of the quotient. These multiplication and division processes are suitable to a long digit computation such as an $\eta$ word precision number where $\eta \geq 2$.

## APPENDIX: SUCESSIVE DIVISION

The recursive computation $s_i$ for $i = 0, 1, 2, \ldots, m$ of the recursive conversion formula can be used for a deterministic division. The deterministic division is explained best by an example.

Consider the division 2518/7 where the radix is decimal, so that $A = 2518$, $B = 7$, $D = 10$ and $P(= D - B) = 3$. The sequential values of $s_i$ computed in decimal are as follows:

$$s_0 = 2, \quad s_1 = 2 \times 3 + 5 = 11, \quad s_2 = 11 \times 3 + 1 = 34, \quad \text{and} \quad s_3 = 34 \times 3 + 8 = 110.$$

The sum $s_0 \times 100 + s_1 \times 10 + s_2 = 344$ can be regarded as a partial quotient and $s_3 = 110$ as the remainder of the division. Since the remainder $s_3 = 110$ is greater than the divisior 7, another computation $s_i$ on 110 must be continued. The second sequential values of $s_i$ are found as follows:

$$s_0 = 1, \quad s_1 = 1 \times 3 + 1 = 4, \quad \text{and} \quad s_2 = 4 \times 3 + 0 = 12.$$

Again, the sum $s_0 \times 10 + s_1 = 14$ can be regarded as a partial quotient and $s_3 = 12$ as the remainder. The third computation of $s_i$ furnishes a partial quotient 1 and the remainder 5. Thus, the total quotient is $344 + 14 + 1 = 359$, and the real remainder is 5.

*Note*: When $P$ is greater than the divisor $B$, a multiple factor $\alpha$ to the divisor may be introduced to accelerate the division (use $\alpha B$ as a pseudodivisor). An obtained remainder of an accelerated division is the remainder of the division $A/(\alpha B)$; therefore, it is necessary to adjust the real remainder of the division $A/B$ from the final result of the accelerated division.

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